IQ Modulation Transfer Functions for a Resonant Cavity

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Introduction: The transfer function expressions for the transmission of small signal amplitude and phase modulations through a resonant cavity are often quoted in the literature without a derivation. The expressions are derived here; both un-normalized and normalized with respect to the steady-state resultant voltage phasor. The un-normalized transfer functions when applied to in-phase (I) and quadrature (Q) components are suitable for large signal analysis of cavity turn-on transients.

In-Phase Component (or small signal amplitude) Modulation:

Assume that a current source is driving a cavity which is represented as a parallel RLC circuit (see Appendix A for the impedance equations and Appendix B for the steady state vector diagram). Furthermore, assume that this current source is amplitude modulated. The amplitude modulation can be expressed as:

$$i(t) = I\left(1 + a\cos\omega_{am}t\right)\cdot\cos\omega_{RF}t \quad (1)$$

where I is considered the un-modulated current magnitude, a is the magnitude of the amplitude modulation at frequency ω_{am}, and ω_{RF} is the RF operating frequency. Using trigonometric identities, (1) can be rewritten as:

$$i(t) = I\cdot\left(\cos\omega_{RF}t \;+\; \frac{a}{2}\cdot\left[\cos(\omega_{RF}+\omega_{am})t + \cos(\omega_{RF}-\omega_{am})t\right]\right). \quad (2)$$

Using phasor notation, the current can be written as $i(t) = \mathrm{Re}\left(\hat{i}\, e^{i\omega_{RF}t}\right)$ where the phasor $\hat{i}$ is expressed as:

$$\hat{i} = I\cdot\left(1 \;+ \frac{a}{2}\left[e^{j\omega_{am}t} + e^{-j\omega_{am}t}\right]\right) \quad (3)$$

with $j = \sqrt{-1}$. Thus, the amplitude modulation expressed in the right hand term of (3) can be represented as two counter-rotating phasors, $e^{j\omega_{am}t}$ and $e^{-j\omega_{am}t}$. These phasors rotate in opposite directions at the modulation frequency.

Using a similar phasor notation, the cavity voltage can be represented as $v(t) = \mathrm{Re}\left(\hat{v}\, e^{j\omega_{RF}t}\right)$. If the driving current, $i(t)$, is applied to the cavity, the resulting voltage phasor is determined via the cavity impedance by:

$$\hat{v} = I\cdot\left(Z(j\omega_{RF}) + \frac{a}{2}\left[Z^{+}e^{j\omega_{am}t} + Z^{-}e^{-j\omega_{am}t}\right]\right) \quad (4)$$

where

$$Z^{+} = Z_{\text{Re}}^{+} + j\,Z_{\text{Im}}^{+} = Z(\,j\omega_{RF} + j\omega_{am}) \quad (5)$$

and

$$Z^{-} = Z_{\text{Re}}^{-} + j\,Z_{\text{Im}}^{-} = Z(\,j\omega_{RF} - j\omega_{am}) \quad (6)$$

The right hand term of (4) relates the transformation of amplitude modulation of the driving current into modulation of the cavity voltage. Using the definition of the complex exponential, the right hand term of (4) can be expressed as:

$$\hat{v} = I\frac{a}{2}\cdot\left\{\left[Z_{\text{Re}}^{+} + j\,Z_{\text{Im}}^{+}\right]\cdot\left[\cos\omega_{am}t + j\sin\omega_{am}t\right] + \left[Z_{\text{Re}}^{-} + j\,Z_{\text{Im}}^{-}\right]\cdot\left[\cos\omega_{am}t - j\sin\omega_{am}t\right]\right\} \quad (7)$$

$$\hat{v} = I\frac{a}{2}\cdot\left[\left\{\left[Z_{\text{Re}}^{+} + Z_{\text{Re}}^{-}\right]\cdot\cos\omega_{am}t - \left[Z_{\text{Im}}^{+} - Z_{\text{Im}}^{-}\right]\cdot\sin\omega_{am}t\right\} + j\cdot\left\{\left[Z_{\text{Im}}^{+} + Z_{\text{Im}}^{-}\right]\cdot\cos\omega_{am}t - \left[Z_{\text{Re}}^{-} - Z_{\text{Re}}^{+}\right]\cdot\sin\omega_{am}t\right\}\right] \quad (8)$$

By making use of the trigonometric identity,

$$A\cos(\omega t + \theta_A) - B\sin(\omega t - \theta_B) = C\cos(\omega t + \theta_C) \quad (9)$$

$$\text{where } C = \sqrt{A^2 + B^2} \quad \text{and } \theta_C = \tan^{-1}\frac{B}{A}\,,$$

the real and imaginary terms of (8) can be condensed resulting in

$$\hat{v} = v_I\cos(\omega_{am}t + \phi_I) + \; j\cdot v_Q\cos(\omega_{am}t + \phi_Q) \quad (10)$$

where

$$v_I = I\frac{a}{2}\left|Z^{+} + Z^{-*}\right|\,,\; \phi_I = \arg\left(Z^{+} + Z^{-*}\right) \quad (11)$$

and

$$v_Q = I\frac{a}{2}\left|Z^{+} - Z^{-*}\right|\,,\; \phi_Q = \arg\left(-j\cdot\left(Z^{+} - Z^{-*}\right)\right) \quad (12)$$

where Z^{-*} denotes the complex conjugate of Z^{-}.

Thus, it is now clear that amplitude modulations of the driving current are transmitted into in-phase and quadrature modulations of the cavity voltage. This process can be described in terms of the following transfer functions:

$$G_{ii}(s) = \frac{V_I(s)}{I_I(s)} \quad \text{and} \quad G_{iq}(s) = \frac{V_Q(s)}{I_I(s)} \quad (13)$$

where $G_{ii}(s)$ denotes transmissions of in-phase driving current modulations to in-phase cavity voltage modulations, $G_{iq}(s)$ denotes transmissions of in-phase driving current modulations to quadruature cavity voltage modulations, and the LaPlace variable, s, is defined in terms of the modulation frequency, $s = i\omega_{am}$.

Using equations, (5), (6), (11), and (12) the transfer functions can be expressed as:

$$G_{ii}(s) = \frac{1}{2}\left[Z(j\omega_{RF} + s) + Z^*(j\omega_{RF} - s)\right] \quad (14)$$

$$G_{iq}(s) = -\frac{j}{2}\left[Z(j\omega_{RF} + s) - Z^*(j\omega_{RF} - s)\right] \quad (15)$$

Normalizing with respect to the un-modulated response, or the steady-state voltage phasor, results in a set of normalized transfer functions:

$$G_{ii}^N(s) = \frac{1}{2}\left[\frac{Z(j\omega_{RF} + s)}{Z(j\omega_{RF})} + \frac{Z^*(j\omega_{RF} - s)}{Z^*(j\omega_{RF})}\right] \quad (16)$$

$$G_{iq}^N(s) = -\frac{j}{2}\left[\frac{Z(j\omega_{RF} + s)}{Z(j\omega_{RF})} - \frac{Z^*(j\omega_{RF} - s)}{Z^*(j\omega_{RF})}\right] \quad (17)$$

The above transfer function expressions can be expanded using the expression for the cavity impedance,

$$Z(s) = \frac{\omega_o \dfrac{R}{Q} \cdot s}{s^2 + \dfrac{\omega_o}{Q} \cdot s + \omega_o^2} = \frac{2\sigma R \cdot s}{s^2 + 2\sigma \cdot s + \omega_o^2} \quad (18)$$

The second expression in (18) utilizes the damping rate parameter, $\sigma = \dfrac{\omega_o}{2Q} = \dfrac{1}{\tau_E}$, which defines the decay rate of the cavity voltage (electric field) impulse response.

However, simpler expressions for the transfer functions can be found if the approximated cavity impedance expression is used. Using,

$$Z(j\omega) \cong \frac{R}{1 - j\dfrac{2Q}{\omega_o}(\omega_o - \omega)} \quad (19) \quad \text{and} \quad Z^*(j\omega) \cong \frac{R}{1 + j\dfrac{2Q}{\omega_o}(\omega_o - \omega)} \quad (20)$$

equations (14)-(17) become:

$$G_{ii}(s)=\frac{\sigma R\cdot(s+\sigma)}{s^2+2\sigma s+\sigma^2(1+\tan^2\phi_Z)} \quad (21)$$

$$G_{iq}(s)=\frac{R\cdot\sigma^2\tan\phi_Z}{s^2+2\sigma s+\sigma^2(1+\tan^2\phi_Z)} \quad (22)$$

$$G_{ii}^N(s)=\frac{\sigma s+\sigma^2(1+\tan^2\phi_Z)}{s^2+2\sigma s+\sigma^2(1+\tan^2\phi_Z)} \quad (23)$$

$$G_{iq}^N(s)=\frac{-\sigma\tan\phi_Z\, s}{s^2+2\sigma s+\sigma^2(1+\tan^2\phi_Z)} \quad (24)$$

Often the expressions for the transfer functions make use of the Hermitian property, $Z(-\omega)=Z^*(\omega)$, of the cavity impedance. Thus, equations (14)-(17) are often written as:

$$G_{ii}(s)=\frac{1}{2}\left[Z(s+j\omega_{RF})+Z(s-j\omega_{RF})\right] \quad (25)$$

$$G_{iq}(s)=-\frac{j}{2}\left[Z(s+j\omega_{RF})-Z(s-j\omega_{RF})\right] \quad (26)$$

$$G_{ii}^N(s)=\frac{1}{2}\left[\frac{Z(s+j\omega_{RF})}{Z(j\omega_{RF})}+\frac{Z(s-j\omega_{RF})}{Z(-j\omega_{RF})}\right] \quad (27)$$

$$G_{iq}^N(s)=-\frac{j}{2}\left[\frac{Z(s+j\omega_{RF})}{Z(j\omega_{RF})}-\frac{Z(s-j\omega_{RF})}{Z(-j\omega_{RF})}\right] \quad (28)$$

However, care must be exercised when using these equations with the approximated cavity impedance function since $Z(-\omega)\neq Z^*(\omega)$ when $Z(\omega)$ is approximated by (19). Thus, if the approximated cavity impedance function of (19) is used, the proper forms of the transfer functions are (14)-(17) while properly substituting for $Z^*(\omega)$ with (20).

Quadrature-Component (or small signal phase) Modulation:

Using a similar process as used for describing amplitude modulations, the transfer functions for quadrature-component modulation can be derived.

Quadrature-component (or small-signal phase) modulation can be expressed as:

$$i_Q(t) = -a_Q \cos\omega_{am} t \cdot \sin\omega_{RF} t \quad (18)$$

The use of the negative sign will become clear from the phasor notation which can be expressed as:

$$i_Q(t) = \mathrm{Re}\left(\hat{i}_Q \, e^{j\omega_{RF} t}\right) \quad (19)$$

with

$$\hat{i}_Q = j \cdot \left(\frac{a_Q}{2} \left[e^{j\omega_{am} t} + e^{-j\omega_{am} t} \right] \right) \quad (20)$$

representing the modulating phasor.

Thus again, the modulating phasor is the sum of two counter-rotating phasors; except this time they are both shifted by 90 degrees. It is the factor of j which has to be accounted for properly to get the right sign for the transfer functions. Again, in general (if $Z^+ \neq Z^{-*}$) the resultant modulation of the cavity voltage has an in-phase and quadrature component.

Again, the resultant cavity voltage modulation phasor can be expressed as:

$$\hat{v} = j \cdot \left(\frac{a_Q}{2} \left[Z^+ e^{j\omega_{am} t} + Z^- e^{-j\omega_{am} t} \right] \right) \quad (21)$$

which becomes similar to (8) except for the factor of j,

$$\hat{v} = j\frac{a}{2} \cdot \left[\left\{ \left[Z^+_{\mathrm{Re}} + Z^-_{\mathrm{Re}} \right] \cdot \cos\omega_{am} t - \left[Z^+_{\mathrm{Im}} - Z^-_{\mathrm{Im}} \right] \cdot \sin\omega_{am} t \right\} + j \cdot \left\{ \left[Z^+_{\mathrm{Im}} + Z^-_{\mathrm{Im}} \right] \cdot \cos\omega_{am} t - \left[Z^-_{\mathrm{Re}} - Z^+_{\mathrm{Re}} \right] \cdot \sin\omega_{am} t \right\} \right] \quad (22)$$

Again, if the resultant is written in terms of in-phase and quadrature components, then it is clear from comparing (22) to (8) that the transfer functions for quadrature-component modulation are identical to those for the in-phase component modulation except for the negative sign introduced by the j^2 factor in the real term resulting from (22).

$$G_{qi}(s) = -G_{iq}(s) \quad \text{and} \quad G_{qq}(s) = G_{ii}(s) \quad (23)$$

Application:

Figure 1 on the right shows a comparison between a reduced model simulation and a full model simulation. Both simulate the turn-on transients of a parallel resonant circuit voltage response to a stepped driving current with a frequency that is unequal to the circuit resonant frequency. The ciruit parameters simulated are $\omega_o = 2\pi \cdot 53 \cdot 10^6$, $\Delta\omega = 2\pi \cdot 20 \cdot 10^3$, $Q = 3500$, and $R/Q = 100$. The amplitude for the reduced model simulation was determined by taking the square root of the sum of the squares of I and Q. The phase was determined by taking the inverse tangent of Q/I.

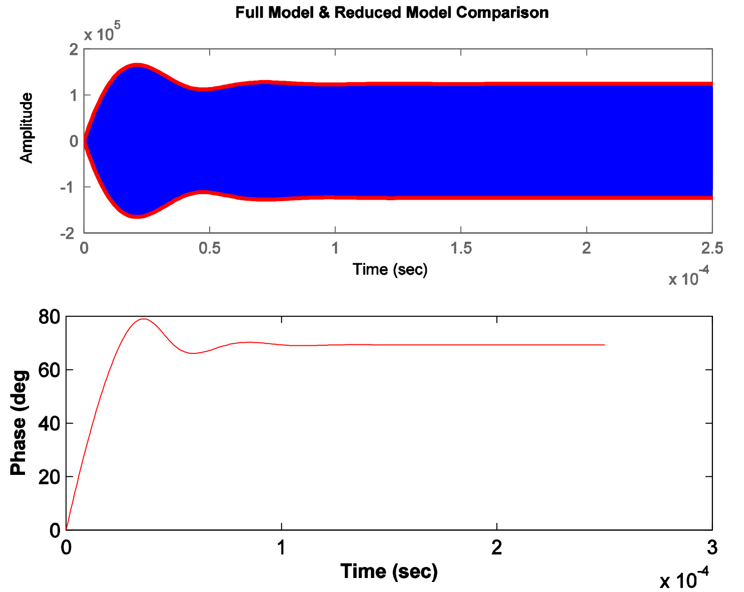


Figure 1: Matlab simulation of the reduced model IQ transfer functions (red traces) compared to a full model simulation (blue trace).

Summary:

Although the transfer functions for IQ modulations in a resonant circuit already exist in the literature, the derivation is not often explained. Furthermore, the literature often states that the transfer function expressions result from the unapproximated cavity impedance; leading one to attempt to reduce the resulting fourth order equation to a second order equation. The expressions in the literature are actually the result of using the approximated cavity impedance for which care needs to be exercised in using the quoted formulas that exploit the Hermitian property of the unapproximated cavity impedance. It is hoped that the derivation presented here will be useful to those studying the transfer functions.

References:

[1] F.Pedersen, "Beam Loading Effects in the CERN PS Booster", IEEE Transactions on Nuclear Science, Vol NS-22, No.3, June 1975, pp 1906-1909.

[2] D.Boussard, "Design of a Ring RF System", CERN Yellow Report 91-04, pp 294-322.

[3] P.B. Wilson, "High Energy Electron Linacs: Applications to Storage Ring RF Systems and Linear Colliders", SLAC-PUB-2884, Nov. 1991.

Appendix A – Cavity Impedance Equations

$Y = \frac{1}{R} + \frac{1}{j\omega L} + j\omega C$ is the admittance of a parallel RLC circuit

$$Z = \frac{1}{Y} = \frac{j\omega RL}{-\omega^2 RLC + j\omega L + R}$$

$$Z = \frac{\frac{1}{C}\cdot j\omega}{-\omega^2 + \frac{1}{RC}\cdot j\omega + \frac{1}{LC}} \quad \text{(A1)}$$

Using the following relations: $\omega_o = \frac{1}{\sqrt{LC}}$ is the natural resonant frequency, and $Q = \omega_o RC$ is the quality factor, the impedance can be expressed as:

$$Z = \frac{\omega_o \frac{R}{Q}\cdot j\omega}{-\omega^2 + \frac{\omega_o}{Q}\cdot j\omega + \omega_o^2} \quad \text{(A2)}$$

$$Z = \frac{R}{1 - i\,Q\left(\frac{\omega_o^2 - \omega^2}{\omega\omega_o}\right)} \quad \text{(A3)}$$

Multiplying numerator and denominator by $1 + i\,Q\left(\frac{\omega_o^2 - \omega^2}{\omega\omega_o}\right)$ gives,

$$Z = \frac{R}{\left[1 + \left(Q\left(\frac{\omega_o^2 - \omega^2}{\omega\omega_o}\right)\right)^2\right]}\cdot\left[1 + i\,Q\left(\frac{\omega_o^2 - \omega^2}{\omega\omega_o}\right)\right] \quad \text{(A4)}$$

The second term can be represented by a complex exponential,

$$1 + i\,Q\left(\frac{\omega_o^2 - \omega^2}{\omega\omega_o}\right) = M\,e^{i\phi_Z} \quad \text{(A5)}$$

where the magnitude, M , is given as

$$M = \sqrt{1 + \left(Q \left(\frac{\omega_o^2 - \omega^2}{\omega \omega_o} \right) \right)^2} = \frac{1}{\cos \phi_Z} \quad \text{(A6)}$$

and the angle, ϕ_Z, which is a function of ω, satisfies

$$\tan \phi_Z = Q \left(\frac{\omega_o^2 - \omega^2}{\omega \omega_o} \right) \cong 2Q \frac{\Delta \omega}{\omega_o} \quad \text{(A7)}$$

where $\Delta \omega \equiv \omega_o - \omega$ and the approximation is found by using the first 2 terms of a Taylor Series expansion in terms of ω.

Thus, the impedance can be expressed as

$$Z = \frac{R}{1 - i \cdot \tan \phi_Z} \quad \text{(A8)}$$

$$Z = R \cos \phi_Z \, e^{i\phi_Z} . \quad \text{(A9)}$$

In terms of the LaPlace variable, $s = j\omega$, the impedance can be written from (A2) as

$$Z(s) = \frac{\omega_o \frac{R}{Q} \cdot s}{s^2 + \frac{\omega_o}{Q} \cdot s + \omega_o^2}, \quad \text{(A10)}$$

Appendix B – Steady-State Vector Diagram

ϕ_S : The synchronous phase angle for below/(above) transition is defined as the +/(-) phase of the beam relative to the positive/(negative) sloped zero-crossing of the RF voltage.

ϕ_B: The beam image current phase angle is defined as the phase of the beam image current relative to the phase of the RF voltage. It is equivalent to -/(+) $\left[\frac{\pi}{2}+\phi_S\right]$ for below/(above) transition. Thus, the beam image current phasor is written as $\hat{I}_B = I_B\, e^{i\phi_B} = I_B\, e^{-/(+)\, i\left[\frac{\pi}{2}+\phi_S\right]}$.

ϕ_Z : The cavity impedance phase angle is defined as the phase of the RF cavity voltage relative to the total cavity drive current. Thus, the total current, beam plus generator, is represented as $\hat{I}_T = I_T\, e^{-i\phi_Z}$

ϕ_L : The load impedance phase angle is defined as the phase of the RF generator current relative to the RF cavity voltage. Thus, the generator current is represented as $\hat{I}_G = I_G\, e^{i\phi_L}$

V_{cav} : The cavity voltage is considered to be at a reference phase of 0.

$I_o \equiv \frac{V_{cav}}{R}$: the generator current required to produce V_{cav} when the cavity is tuned to resonance (when $\phi_Z = 0$)

Thus, to make V_{cav} when the cavity is driven off resonance:

$$\hat{I}_T = \frac{V_{cav}}{Z} = \frac{V_{cav}}{R\cos\phi_Z}\, e^{-i\phi_Z} = \frac{I_o}{\cos\phi_Z}\, e^{-i\phi_Z} \qquad \text{(B1)}$$

Now, since the total current is the sum of the generator plus beam image currents:

$$I_G\, e^{i\phi_L} + I_B\, e^{-/(+)i\left[\frac{\pi}{2}+\phi_S\right]} = \frac{I_o}{\cos\phi_Z} e^{-i\phi_Z} \quad \text{(B2)}$$

separating into real and imaginary components:

Real components: $I_G \cos\phi_L - I_B \sin\phi_S = I_o$ (B3)

Imaginary components: $I_G \sin\phi_L -/(+)\; I_B \cos\phi_S = -I_o \tan\phi_Z$ (B4)

Thus, given I_B, ϕ_S, and V_{cav} and using ϕ_L as a free parameter, the generator current, I_G, and cavity impedance angle, ϕ_Z, can be determined. Alternatively, one can use ϕ_Z as the free parameter and thus determine I_G and ϕ_L.

Using ϕ_L as the free parameter:

From (B3) we obtain: $I_G = \dfrac{I_o + I_B \sin\phi_S}{\cos\phi_L} = \dfrac{I_o\left(1 + Y \sin\phi_S\right)}{\cos\phi_L}$ (B5)

Then from (B4) we obtain:

$$\tan\phi_Z = \frac{-I_G \sin\phi_L +/(-)\; I_B \cos\phi_S}{I_o} = -\left(1 + Y\sin\phi_S\right)\tan\phi_L +/(-)\, Y\cos\phi_S \quad \text{(B6)}$$

where $Y \equiv \dfrac{I_B}{I_o}$ (B7) is called the beam loading factor

Using ϕ_Z as the free parameter:

Solving (B4) for $I_G \sin\phi_L$ and then dividing this by $I_G \cos\phi_L$ from (B3):

$$\tan\phi_L = \frac{-\tan\phi_Z +/(-)\; Y\cos\phi_S}{\left(1 + Y\sin\phi_S\right)} \quad \text{(B8)}$$

and then using (B5)

$$I_G = \frac{I_o\left(1 + Y\sin\phi_S\right)}{\cos\phi_L} \quad \text{(B9)}$$

Cavity Vector Diagram

V_{cav}: cavity voltage

I_G: generator current

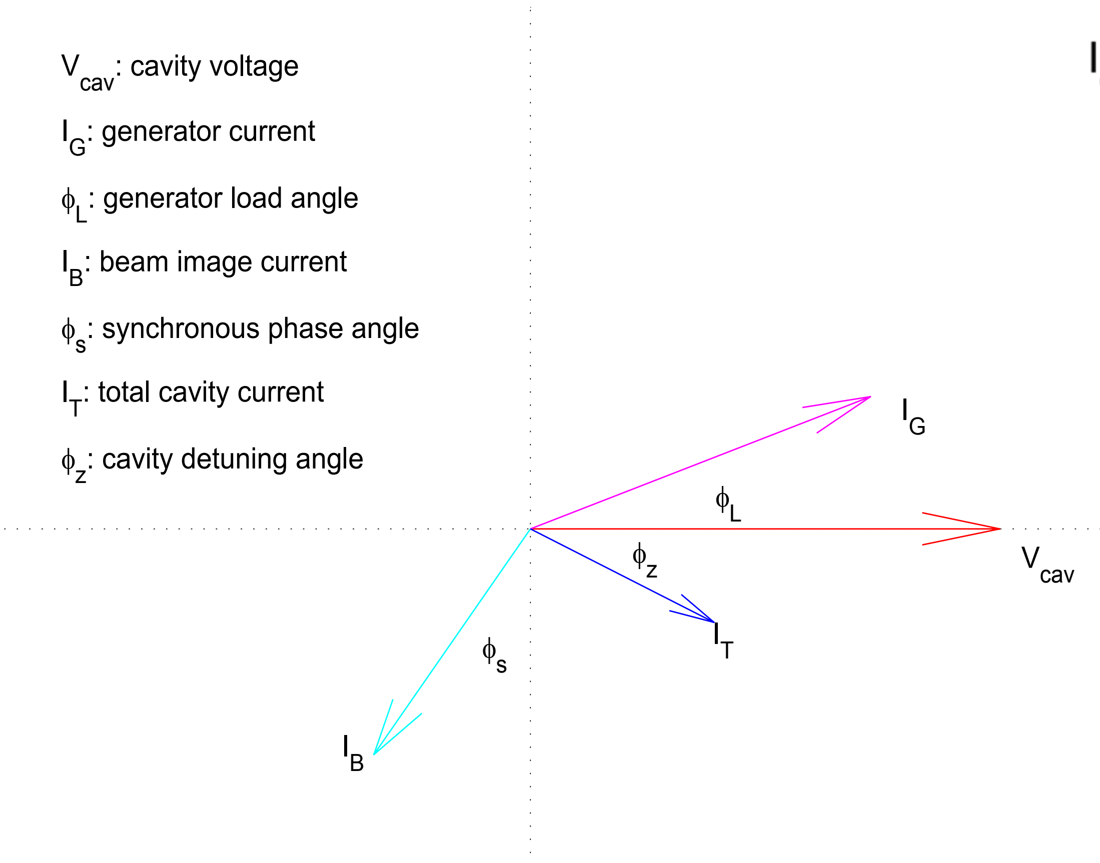
ϕ_L: generator load angle

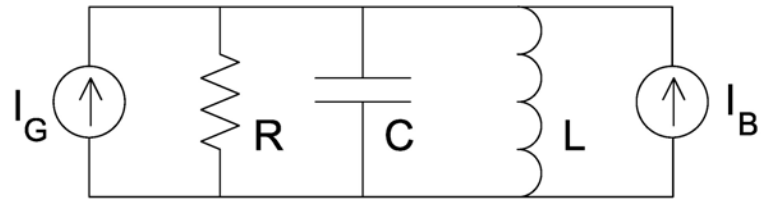
I_B: beam image current

ϕ_s: synchronous phase angle

I_T: total cavity current

ϕ_z: cavity detuning angle





$$Z(\omega) = R\cos\phi_Z \cdot e^{i\phi_Z}$$

$$\tan\phi_Z = Q\left(\frac{\omega_o^2 - \omega^2}{\omega\omega_o}\right) \cong 2Q\frac{\Delta\omega}{\omega_o} \qquad \Delta\omega \equiv \omega_o - \omega$$

$$\hat{V}_{Cav} = \hat{I}_T \cdot Z \qquad \hat{I}_T = \hat{I}_G + \hat{I}_B$$

$$I_o \equiv \frac{V_{Cav}}{R} \text{ :current to make } V_{Cav} \text{ when } \phi_Z = 0$$

$$I_G\, e^{i\phi_L} + I_B\, e^{-/(+)i\left[\frac{\pi}{2}+\phi_S\right]} = \frac{I_o}{\cos\phi_Z} e^{-i\phi_Z}$$

$$I_G = \frac{I_o(1 + Y\sin\phi_S)}{\cos\phi_L} \qquad Y \equiv \frac{I_B}{I_o}$$

$$\tan\phi_Z = -(1 + Y\sin\phi_S)\tan\phi_L +/(-)\, Y\cos\phi_S$$

$$\tan\phi_L = \frac{-\tan\phi_Z +/(-)\; Y\cos\phi_S}{(1 + Y\sin\phi_S)}$$